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LETTERS TO THE EDITOR

EXACT SOLUTIONS FOR VIBRATIONS OF SOME ANNULAR MEMBRANES WITH INHOMOGENEOUS RADIAL DENSITIES

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1. INTRODUCTION

In a recent paper, Wang [1] found some closed-form exact solutions for the vibrations of two types of continuously non-homogeneous membranes, including the case of the complete vibration spectrum of an annular membrane with density proportional to the inverse square of the radius.

The spatial eigenfunctions for this latter case, with $\rho \propto 1/r^2$, for an annulus of inner radius a and unit outer radius, were shown in reference [1] to be of the form

$$\cos(n\theta)\sin[\sqrt{K^2 - n^2}\ln(r)]. \tag{1}$$

where $n = 0, 1, 2, \ldots$ is the angular index. K is a dimensional frequency parameter given explicitly by

$$K = \sqrt{\left(\frac{m\pi}{\ln(a)}\right)^2 + n^2}, \tag{2}$$

where $m = 1, 2, 3, \ldots$ is the radial index. The gravest frequency ($n = 0, m = 1$), with a certain normalization specified in reference [1], was tabulated there for a range of radius ratios a, $0 \leqslant a < 1$.

Actually, equation (2) for the frequency parameter was already obtained by De [2] in 1971, together with, implicitly, equation (1) for the eigenfunctions. The radially symmetric case $n = 0$ is interesting in that it results in a harmonic frequency spectrum for the radial modes, with $K \propto m$. The frequencies and mode functions for this case have been known for a very long time, e.g., reference [3]. They may also readily be obtained from the constant-density string equation by a logarithmic transformation of the co-ordinate [4].

The purpose of this note is to draw attention to some other exact solutions for vibrations of annular membranes with inhomogeneous densities, both continuously varying and continuously varying with a step discontinuity. In terms of plane polar co-ordinates r, θ, the equation to be solved for the radial part of the displacement $u(r)$ is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{n^2}{r^2}u = -\frac{\rho(r)}{\tau}\omega^2 u \tag{3}$$

together with the boundary conditions of fixed edges, where the angular function may be taken as $\cos(n\theta)$, $n = 0, 1, 2, \ldots$, and ω is the modal radian frequency of vibration. Here, $\rho(r)$

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is the radius-dependent density for the inhomogeneous membrane, and τ is the tension parameter, henceforth set equal to 1. In some cases it is possible to obtain the complete spectrum in simple explicit analytical form; in other cases, it may be only the radial modes.

From equation (3) it may be seen that only in the case of inverse square radial dependence

$$\rho(r) = \alpha/r^2, \tag{4}$$

with α constant, are the frequencies for asymmetrical motions $n \neq 0$ immediately obtainable from the radially symmetrical modes by the simple replacement of ω^2 by $(\omega^2 - n^2/\alpha)$ (as in equation (2) and similarly for the modal functions which remain unchanged in their functional dependence on r (cf. equation (1)).

2. INVERSE FOURTH POWER RADIAL DENSITY

It was shown in reference [5] that an inhomogeneous annular membrane ($R_1 \leqslant r \leqslant R_2$) with inverse fourth power radial density

$$\rho(r) = \rho_0 \frac{R_1^2 R_2^2}{r^4} \tag{5}$$

has exactly the same *complete* vibration spectrum for all the modes as the congruent *homogeneous* annular membrane with constant density ρ_0. The eigenfunctions for the inhomogeneous case are obtained from those of the homogeneous case, involving Bessel functions of the first and second kinds for the radial part and trigonometric functions for the angular part, by replacing the radius variable r by $R_1 R_2/r$ and leaving θ unchanged.

This exact complete solution for this inhomogeneous annular membrane thus supplements that of reference [1] for the inverse square case. It also shows that a complete vibration spectrum does not uniquely determine the density structure of an annular membrane.

3. INVERSE SQUARE WITH FOURTH POWER LOGARITHM RADIAL DENSITY

The radial (i.e., axisymmetric) modes of vibration of an annular membrane whose density contains not only an inverse square radial term as in section 1 above but also an inverse fourth power logarithmic term were solved for exactly in reference [4]. The membrane occupies the annulus $R_1 < r < R_2$, with $a = R_1/R_2 < 1$. The density here is given by

$$\rho(r) = \frac{c}{r^2} \frac{1}{[1 + \gamma \ln(R_2/r)]^4}, \tag{6}$$

where γ is a "distortion parameter" corresponding to variation from the simple inverse square form. The normalization criterion for density (6) is now presented to conform with reference [1], i.e., the averaged density over the annulus area is here taken as unity:

$$2\pi \int \rho(r) r \, \mathrm{d}r = \pi(R_2^2 - R_1^2), \tag{7}$$

so that here

$$c = \frac{3\gamma}{2} R_2^2 (1 - a^2) \frac{(1 + \gamma |\ln a|)^3}{(1 + \gamma |\ln a|)^3 - 1}. \tag{8}$$

With some change in notation from reference [4], the radial function is given explicitly by

$$u_m(r) = [1 + \gamma \ln(R_2/r) \sin\left[m\pi \frac{1 + \gamma|\ln a|}{|\ln a|} \frac{\ln(R_2/r)}{1 + \gamma \ln(R_2/r)} \right]. \tag{9}$$

The radial spectrum is harmonic ($m = 1, 2, 3, \ldots$) and takes the explicit values

$$\omega_m = \frac{m\pi}{R_2} \sqrt{\frac{2}{3}} \sqrt{\frac{(1 + \gamma|\ln a|)^3 - 1}{(1 - a^2)(\ln a)^2\, \gamma(1 + \gamma|\ln a|)}}. \tag{10}$$

It is of interest to compare the overall gravest mode frequency of this annular membrane, which is the lowest frequency of the radial spectrum, i.e., equation (10) with $m = 1$, with the gravest mode of the simple inverse square density annular membrane of Wang [1] normalized similarly; henceforth, $R_2 = 1$ is taken. First of all, it should be noticed that as the distortion parameter $\gamma \to 0$ (with inner radius $a \neq 0$), equation (10) reduces to

$$\omega_1|_{\gamma\to 0} = \pi \sqrt{\frac{2}{(1 - a^2)|\ln a|}} \tag{11}$$

which is just Wang's expression (equation (15) in reference [1]) for the gravest case $n = 0$, $m = 1$ there, as desired. On the other hand (for distortion parameter $\gamma \neq 0$), as inner radius $a \to 0$ there results

$$\omega_1|_{a\to 0} = \pi \sqrt{\tfrac{2}{3}} \sqrt{\gamma} \tag{12}$$

which is non-zero. This curious result, which is unphysical insofar as the actual density becomes singular at the origin as the annulus becomes a circle, is discussed in Appendix A. In any case, if both a and γ tend to zero in either order, then by equations (11) and (12), the frequency approaches zero in agreement with the $a = 0$ case of Table 2 in reference [1] for the pure inverse square density membrane.

Values of the fundamental angular frequency parameter (equation (10) with $m = 1$) are tabulated in Table 1 for various values of a and γ for comparison with the values for the pure inverse square density annular membrane ($\gamma = 0$) as obtained in Table 2 of reference [1] in addition to the homogeneous case. It is seen that, as for the case $\gamma = 0$ of reference [1],

TABLE 1

Fundamental angular frequency ω_1 (eq. (10), $m = 1$) of annular membrane, inner radius a, outer radius $R_2 = 1$, with radial density equations (6), (8); γ = distortion parameter

		γ					
a	HOMOG[1)]	0[2)]	0·001	0·01	0·1	1	10
0[3]	2·405	0	0·08112	0·2565	0·8112	2·565	8·112
0·001	2·654	1·690	1·690	1·692	1·768	2·933	8·229
0·01	2·801	2·070	2·070	2·071	2·120	3·113	8·288
0·1	3·314	2·943	2·943	2·943	2·964	3·646	8·506
0·5	6·246	6·162	6·162	6·162	6·167	6·447	10·707
0·9	31·412	31·401	31·401	31·401	31·402	31·454	34·113

[1]Ref. [11].
[2]Equation (11); cf reference [1].
[3]Equation (12), for inhomogeneous membrane.

the frequency for fixed $\gamma > 0$ increases as the inner radius a increases. However, for fixed a, the frequency also increases as γ increases from zero, and (unlike the $\gamma = 0$ case [1]) it exceeds the homogeneous value by the time $\gamma = 1$.

4. SOME STEPPED CASES

4.1. STEPPED INVERSE SQUARE RADIAL DENSITY

In reference [6] it was shown that for a stepped inverse square density distribution

$$\rho(r) = d_1/r^2, \quad R_1 \leqslant r < R_3, \qquad \rho(r) = d_2/r^2, \quad R_3 \leqslant r \leqslant R_2, \tag{13}$$

the radial vibration spectrum is purely harmonic, as for the continuous case in section 1 above, provided that the constants d_1 and d_2 satisfy the relation

$$(d_1/d_2)^{1/2} = [\log(R_2/R_3)/\log(R_3/R_1)]. \tag{14}$$

The radial eigenfrequencies (for unit tension) are given by

$$\omega_m = m\pi/(2d_1^{1/2}\log(R_3/R_1)). \tag{15}$$

The eigenfunctions again consist of sine functions with logarithms of r in their arguments [6].

This exact solution for the special stepped case complements the continuous case of section 1, for the radial modes. However, the asymmetric modes are now not obtained by a simple replacement of ω as in the continuous case mentioned at the end of section 1, because the coefficient of $1/r^2$ in equation (13) is not constant throughout the membrane. An explicit analytical expression for the frequencies of the angle-dependent modes is not attainable in this instance, although they may be obtained numerically from the appropriate characteristics equation for chosen values of d_i.

4.2. STEPPED INVERSE SQUARE WITH FOURTH POWER LOGARITHM DENSITY

The ideas of sections 3 and 4.1 can be combined to given another new exactly solvable case: the radial vibrations of an inhomogeneous membrane with a stepped density which contains inverse square and fourth power logarithm terms in the density function. The radial spectrum is again harmonic for a specific parameter ratio relation.

Let

$$\rho(r) = \rho_1(r), \quad R_1 \leqslant r < R_3, \qquad \rho(r) = \rho_2(r), \quad R_3 \leqslant r \leqslant R_2, \tag{16a}$$

where

$$\rho_i(r) = \frac{D_i}{r^2[1 + \Gamma\log(r/R_1)]^4}, \tag{16b}$$

where D_1, D_2 and Γ are constants. The radial functions in each part of the membrane have functional dependence on r similar in form to equation (9). Then, provided that

$$(D_1/D_2)^{1/2} = \frac{1}{1 + \Gamma\log(R_2/R_1)}[\log(R_2/R_3)/\log(R_3/R_1)], \tag{17}$$

the radial spectrum is found to be exactly pure harmonic: explicitly

$$\omega_m = m\pi[1 + \Gamma \log(R_3/R_1)]/(2D_1^{1/2} \log(R_3/R_1)). \tag{18}$$

5. DISCUSSION

Closed-form exact solutions to vibration problems may be of two types. First, the exact eigenfunctions are known explicitly in terms of standard functions and involve the frequency parameter which satisfies an exact explicit characteristic equation which must be solved numerically. Secondly, there is *also* an exact explicit analytical formula for the frequencies themselves in terms of the system's physical parameters.

With regard to reference [1] and this paper, the first category includes the rectangular membrane with a linear density variation as discussed in reference [1], and the following cases presented above: inverse fourth power radial density, section 2, for all modes; stepped inverse square radial density as in section 4.1, for all modes and parameter ratios; and section 4.2 (radial modes) for general parameter ratios. The second category includes the inverse square radial density annular membrane of reference [1] and section 1 above, for all modes, and other cases in the present paper: inverse square with fourth power logarithm radial density, section 3, radial modes; section 4.1, radial modes, when relation (14) is satisfied; and section 4.2, radial modes, when relation (17) is satisfied.

This second category is especially useful for verifying computational methods and algorithms.

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APPENDIX A: A CIRCULAR CASE

The finite positive explicit limit, equation (12), for the fundamental frequency of the circular limit $a \to 0$ of the annulus with density given by equation (6) may be understood as follows. Setting outer radius $R_2 = 1$ for convenience, the density is

$$\rho(r) = \frac{c}{r^2} \frac{1}{[1 + \gamma \ln(1/r)]^4}. \tag{A1}$$

Even though this becomes infinite as r tends to zero, the mean density over the circular region is still unity, i.e., finite, in that, for $\gamma \neq 0$,

$$2\int_0^1 \rho r\, dr = 1, \qquad c = (3/2)\gamma. \tag{A2a, b}$$

Then the radial function is

$$v_m(r) = [1 + \gamma \ln(1/r)] \sin\left[m\pi\gamma \frac{\ln(1/r)}{1 + \gamma \ln(1/r)}\right]. \tag{A3}$$

It satisfies the fixed boundary condition at $r = 1$ and the Helmholtz equation (3) (with $n = 0$ and $\tau = 1$), with

$$\omega_m = m\pi\sqrt{\tfrac{2}{3}}\sqrt{\gamma}, \tag{A4}$$

in agreement with equation (12), even though $v_m(r)$ itself becomes infinite at the centre of the circle, i.e., as $r \to 0$. Thus the mean density may be normalized and the frequencies maintain finite values even though the density and radial function become unphysical as r approaches the origin of the circle.

By contrast, for the pure inverse square density case, $\gamma = 0$ in equation (A1), the density cannot be normalized over a complete circular region to have mean value 1 as in equation (A2a).